ARE THERE IMPORTANT UNITARITY CORRECTIONS TO THE ISOBAR MODEL?

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Recently developed formalism is used to calculate rescattering effects in the isobar model. We evaluate these corrections for the reaction $N\pi \to N\pi\pi$ up to 1.5 GeV to assess their influence on existing phenomenology. Our results also include the identification of distinct kinds of subenergy variation, each physically interpretable. Interesting features in the total energy dependence are also observed.

The isobar model [1] has been used extensively to analyse three-hadron final states [2,3]. In the model the production amplitude takes the form of a sum over all isobars in each of the three final state isobar channels [‡1]

$$M(s, s_1, s_2, s_3) = \sum_{\alpha=1}^{3} \Omega_\alpha k_\alpha M_\alpha(s_i) f_\alpha \hat{M}(s) , \qquad (1)$$

in which $s = W^2$ and $s_i = w_i^2$ for total energy W and subenergy w_i. Ω_α and k_α stand for the requisite angular and kinematic factors. M_α and $\hat{M}$ are the elastic two-body amplitudes with isobar and initial state quantum numbers, respectively. The quantity of interest is the isobar factor f_α; in the phenomenological applications [2,3] it is assumed to be s_i-independent. It was first clearly realized by Aaron and Amado [4] that this feature of the model violates unitarity in the two-body subenergy variables. Because of this the existing phenomenology is open to question, especially for those results in which the phases of amplitudes are among the extracted information. Formalism which we have recently developed [5–8] implements subenergy unitarity and analyticity in the isobar framework, providing for an improved phenomenology. Thus, calculations bearing on the question raised in the title of this paper can now be carried out. We summarize the results below, for the reaction $N\pi \to N\pi\pi$ at energies up to $W = 1.5$ GeV.

The s_i-dependence of f_α in eq. (1) follows from the discontinuity relations obtained from subenergy unitarity [6]; these are implemented analytically by the use of dispersion relations. The resulting expressions for each J^P wave can be manipulated [7,9] into single-variable integral equations [8]:

$$f_\alpha(s, s_i) = c_\alpha(s, s_i) + \sum_{\substack{\beta \\ j\neq i}} \int_{-\infty}^{\hat{z}_j} K_{\alpha\beta}(s, s_i, z_j) f_\beta(s, z_j)\, \mathrm{d}z_j . \qquad (2)$$

The inhomogeneous term c_α has no unitarity cut in s_i

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[‡1] In the $N\pi\pi$ final state, the $N\pi$ isobars occur with subenergy variables s_1 and s_2, and the $\pi\pi$ isobars with s_3. Of course the amplitude must be Bose symmetric in the two pions.

and is otherwise arbitrary. For phenomenological purposes it may be taken to be independent of s_i, so that it can be identified with the isobar factor of the conventional (non-unitary) model; its unitarization arises from the integral term in eq. (2). The kernel is a known function; apart from the necessary kinematical encumbrances, it contains a basic dispersion integral of the sort evaluated in ref. [7, Appendix B] and a unitary two-body function describing $M_\beta(z_j)$. The upper limit of integration in eq. (2) is at the upper edge of the Dalitz plot.

The Fredholm solution of eq. (2) would provide a unitary analytic phenomenology; however, for our purposes the formidable task of inverting eq. (2) can be by-passed. We wish to assess the extent to which unitarity induces substantial subenergy dependence in f_α, in view of the impact this might have on the validity of the existing isobar model fits. To this end we adopt the basic phenomenological form:

$$f_\alpha(s, s_i) = c_\alpha(s) + \sum_\beta I_{\alpha\beta}(s, s_i)\, \bar{c}_\beta(s)\,. \tag{3}$$

To get this we argue that the major s_i-variation is obtained by a first iteration of eq. (2), in which we replace f_β by c_β. This argument is supported by numerical calculations in the 3π system [10], even in circumstances where rescattering effects are strong enough to generate a three-body resonance. The latter behavior corresponds to s-dependence and is allowed for by the presence in eq. (3) of the new quantity $\bar{c}_\beta$, which effectively absorbs the (uncalculated) s-dependence provided by the Fredholm denominator. Thus c_α and $\bar{c}_\beta$ comprise a new set of phenomenological fitting parameters. When we consider the origin of $\bar{c}_\beta$ we might reasonably estimate it to be of the same order of magnitude as c_β unless there are dynamical circumstances, such as three-body resonance behavior, whereby the Fredholm denominator might develop substantial s-dependence; in this case $\bar{c}_\beta$ would be enhanced over c_β. The primary focus of our calculations in each J^P wave is the *rescattering integral:*

$$I_{\alpha\beta}(s, s_i) = \int_0^{\hat{z}_j} K_{\alpha\beta}(s, s_i, z_j)\, \mathrm{d}z_j\,. \tag{4}$$

The argument by which eq. (3) accounts for the dominant s- and s_i-dependence is based on the dominance of nearby singularities, already contained in the first iteration. It is consistent with this approach that we have truncated the rescattering integral (4) at $z_j = 0$. We have performed several calculations, varying s and s_i, to justify this truncation. The choice of different cutoffs always produces results which differ by amounts having negligible dependence on s_i. Clearly the contributions to the solution which lack s_i-variation are readily absorbed into the fitting parameters c_α and $\bar{c}_\beta$. Eq. (3) is shown graphically in fig. 1, in which isobars produced in the state β are rescattered to produce isobars in the final state α.

We have calculated $I_{\alpha\beta}$ in the $N\pi\pi$ system for energies up to $W = 1.5$ GeV. The isobars we retain are: all s-wave isobars ($N\pi$ S_{11} and S_{31}, $\pi\pi$ isospin 0 and 2) in S-states, active p-wave isobars ($N\pi$ P_{11} and P_{33}, $\pi\pi$ isospin 1) in S-states, and the $N\pi$ P_{33} isobar in P-states. The J^P waves are $1/2^+$, $1/2^-$, $3/2^+$ and $3/2^-$. The unitary two-body functions appearing in $K_{\alpha\beta}$ in eq. (4) are constructed to fit known $N\pi$ data [11] and $\pi\pi$ data [12].

The first and perhaps most important result is that *no* rescattering integral $I_{\alpha\beta}$ shows any really rapid subenergy variation for $W \lesssim 1.5$ GeV. In almost all cases the shape of the s_i-variation is very much the same for different values of s. This implies that, although there may be some observable corrections to the subenergy spectra, true s-channel resonance behavior will not be seriously distorted. Thus our calculations provide a measure of retrospective justification for the use of the non-unitary isobar model in the $N\pi\pi$ system at these energies. It would appear that the question raised in our title has been answered in the negative. Despite this, there remain several notable features in the subenergy dependence of our calculated $I_{\alpha\beta}$ [‡2], some of which may have detectable effects.

‡2 Our notation for the isobars in channels α and β is: S_1, S_3, Δ and $\mathcal{N}$ for the $N\pi$ isobars S_{11}, S_{31}, P_{33} and P_{11}; and ϵ_0, ϵ_2 and ρ for the $\pi\pi$ isobars with isospin 0, 2 and 1. No isospin-designating subscript is called for in channel α because the isospin of α is irrelevant. Finally, $I_{\alpha\beta}$ bears a J^P superscript.

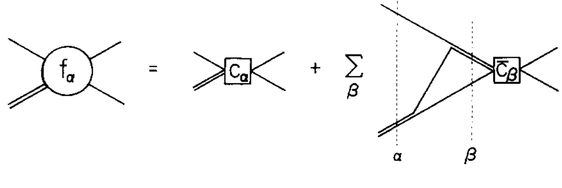

Fig. 1. Rescattering effects in the isobar model.

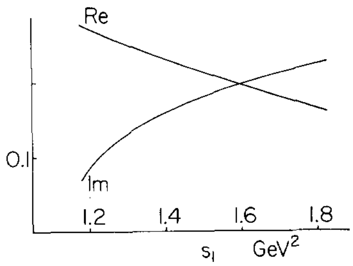

Fig. 2. Rescattering integral $I_{SS_1}^{1/2+}$ for W = 1.5 GeV, showing behavior characteristic of the scattering length effect.

Our principal finding is that the calculated subenergy variations fall into a number of well defined classes. One class of integrals shows s_i-variation which depends quite markedly on s; this is due to a logarithmic singularity [13]. In all other cases the s-variation is slight, and the types of s_i-variation we find are all closely correlated with the orbital angular momentum quantum numbers of the states involved. We shall postpone the logarithmic singularity effects for the moment and begin with the simplest configurations, in which all angular momenta are zero. These occur in I_{SS_1}, I_{SS_3}, $I_{S\epsilon_0}$, $I_{S\epsilon_2}$, $I_{\epsilon S_1}$ and $I_{\epsilon S_3}$, each having $J^P = 1/2^+$. The first of these is plotted in fig. 2. It is remarkable that all of the others exhibit the same general features as shown in fig. 2: namely, a curvature at the subenergy threshold and a striking crossover of the real and imaginary parts of $I_{\alpha\beta}$. We have found that this kind of behavior is well represented by the expression

$$I_{\alpha\beta}(s, s_i) = c_1(s) + c_2(s)\, g(q_i)\,, \tag{5}$$

where g has the "scattering length" form

$$g(q_i) = (1 - \mathrm{i}aq_i)^{-1}\,, \tag{6}$$

in which q_i is the momentum of one of the two particles in the rest frame of isobar i. Obviously c_1 and c_2 are without significance since they are absorbed in c_α and $\bar{c}_\beta$ in eq. (3). The values of the parameter a are listed in table 1 for a rough fit at W = 1.5 GeV. The numbers are physically reasonable and in all cases *positive.* It should be noted that eq. (5) realizes a proposal of Aaron and Amado [4] for the production of s-wave isobars, and that the threshold enhancement produced by eq. (6) is similar in nature to a well-established effect in nd → nnp at low energy [14]. We would expect to see this effect whenever we have s-wave isobars produced in S-states.

Table 1
Scattering length parameter (see eq. (6)) for W = 1.5 GeV.

	$J^P = 1/2^+$					
	I_{SS_1}	I_{SS_3}	$I_{S\epsilon_0}$	$I_{S\epsilon_2}$	$I_{\epsilon S_1}$	$I_{\epsilon S_3}$
a (fm)	0.3	0.6	0.8	0.3	1.0	0.5

Another clear type of subenergy variation to be observed is one in which the real and/or the imaginary parts of $I_{\alpha\beta}$ show curvature toward the upper end of phase space; moreover, when both parts are varying they tend to do so with approximately parallel s_i-dependence. The integrals $I_{\Delta\Delta}$ for $J^P = 1/2^+$ and $3/2^+$ manifest this; $I_{\Delta\Delta}^{1/2+}$ is plotted in fig. 3. For both of these cases the isobar Δ is in a P-state in both channels α and β. Since the observed curvature occurs where the isobar momentum in state α $Q_\alpha \to 0$, we are led to consider an angular momentum barrier interpretation. General considerations [15] would suggest the occurrence of a range-dependent barrier factor in the intensity, having the form:

$$b(Q_\alpha) = (RQ_\alpha)^{2L_\alpha}/[1 + (RQ_\alpha)^{2L_\alpha}]\,. \tag{7}$$

The kinematic factors in eq. (1) already provide the numerator of eq. (7), so we might therefore expect the isobar factor f_α to exhibit a "barrier" dependence:

$$h(Q_\alpha) = \mathrm{e}^{\mathrm{i}\theta(Q_\alpha)}/[1 + (RQ_\alpha)^{2L_\alpha}]^{1/2}\,. \tag{8}$$

Such a barrier feature produces just the curvature near $Q_\alpha = 0$ which we wish to interpret. A rough fit to these two cases has been performed using:

$$I_{\alpha\beta}(s, s_i) = c_1(s) + c_2(s)\, h(Q_\alpha)\,. \tag{9}$$

Because of the approximately parellel behavior of the real and imaginary parts, θ in eq. (8) is approximately

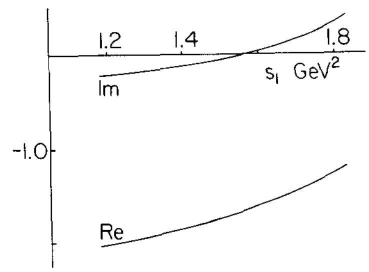

Fig. 3. Rescattering integral $I_{\Delta\Delta}^{1/2+}$ for W = 1.5 GeV, showing behavior characteristic of the barrier effect.

Table 2
Barrier parameter (see eq. (8)) for $W = 1.5$ GeV.

	$I^{1/2+}_{\Delta S_1}$	$I^{1/2+}_{\Delta\Delta}$	$I^{1/2+}_{\Delta\epsilon_2}$	$I^{3/2+}_{\Delta\Delta}$
R (fm)	0.4	1.2–1.6	1.2	1.2–1.6

independent of Q_α; consequently only the real parameter R has any significance. The values of R at $W = 1.5$ GeV for these and several other $I_{\alpha\beta}$ are listed in table 2. The numbers are physically reasonable, so that here too we have obtained a satisfactory interpretation of the results which we conjecture is likely to be general.

A case intermediate, in angular momentum terms, between the previous two is that involving p-wave isobars in S-states. Examples in our calculations are $I^{1/2-}_{\rho\mathcal{N}}$, $I^{3/2-}_{\rho\Delta}$, $I^{3/2-}_{\Delta\Delta}$, $I^{1/2-}_{\mathcal{N}\mathcal{N}}$, $I^{1/2-}_{\mathcal{N}\rho}$ and $I^{3/2-}_{\Delta\rho}$. The latter two clearly show the logarithmic singularity (see below), but the rest all have approximately linear behavior in s_i. Such a variation corroborates a proposal of Aaron and Amado [4] concerning the production of p-wave isobars, and is expected to contribute only a small effect.

Evidence can be found among our results for the kind of structure which can be identified with the logarithmic singularity of the triangle graph [13]. This effect is expected when the state β in fig. 1 contains a well-defined resonance. A peak shows up near the subenergy threshold, and does so in a manner which changes markedly with W. This feature is due to the migration of the triangle singularity as W varies through an energy region near the resonance-plus-third-particle mass in channel β. For $W \leqslant 1.5$ GeV, only the Δ isobar in state β fulfills this criterion, and of the possible candidates likely to show the effect only $I^{1/2+}_{\epsilon\Delta}$ does so clearly; this phenomenon is shown in fig. 4. No doubt kinematic factors suppress the effect in the other cases for which the state β contains the Δ. It should be noted that when β contains the ρ the same circumstances for the logarithmic singularity should arise, but for W near 1.7–1.8 GeV. Indeed this expectation is confirmed by the rescattering integral $I^{3/2-}_{\Delta\rho}$ $(= I^{1/2-}_{\mathcal{N}\rho}/\sqrt{2})$, as shown in fig. 5.

It is interesting that the $I_{\alpha\beta}$'s which vary with s_i and have the greatest magnitudes are those for which α and β contain one of ρ, ϵ_0 and Δ, precisely the isobar channels retained in the non-unitary analyses [2]. In this context we should emphasize, however, that the contribution of the isobar $\mathcal{N}$ should not be left out of any future phenomenology. Even though we have found the $\mathcal{N} \to \mathcal{N}$ rescattering to have only slight s_1-variation, the effect of the active $\mathcal{N}$ isobar via the factor M_α in eq. (1) ought not to be so negligible. It is unfortunate that the scattering length effect (eqs. (5), (6) and fig. 2) is associated with isobars in α and/or β which have not been included in the successful standard analyses; these must therefore be small-magnitude waves. Detection of the barrier effect (eqs. (8), (9) and fig. 3) is more promising since it arises in Δ production. The $\Delta \leftarrow \Delta$ rescattering is substantial in magnitude and in s_1-variation. We have predicted $R \approx 1.4$

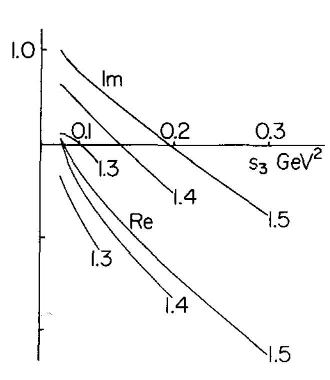

Fig. 4. Rescattering integral $I^{1/2+}_{\epsilon\Delta}$ for $W = 1.3$, 1.4 and 1.5 GeV, showing behavior characteristic of the triangle singularity effect.

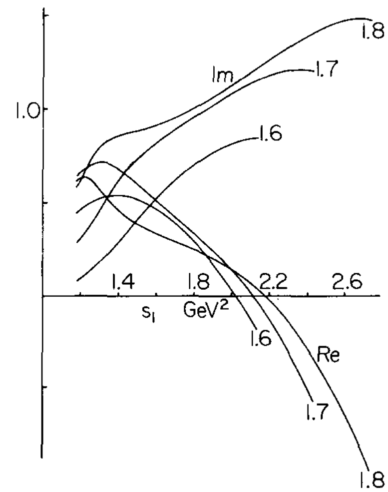

Fig. 5. Rescattering integral $I^{3/2-}_{\Delta\rho}$ for $W = 1.6$, 1.7 and 1.8 GeV, showing the effect of the triangle singularity.

fm at $W = 1.5$ GeV, and this effect should be observable with improved data near the $\pi\Delta$ threshold.

Full details of the calculations and results summarized here will be published in a separate paper [16].

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